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Note

Two-Point Quasi-fractional Approximations to the Bessel Functions $J_\nu(x)$ of Fractional Order

I. Introduction

Bessel functions of fractional order appear in several areas of physics. Although the power series for these functions can be used for any finite value of the variable, their computation is not easy since many terms of the power series have to be taken for even small values of x. On the other hand, most of the approximations are either valid only piecewisely in intervals of the variable x or limited to a given order [1–5]. In this paper we give two approximations for the Bessel functions in such a way that the complete range of ν between -1 and 1 is covered and they are valid for the full range of the variable $x \geqslant 0$. As in previous papers, the parameters of the quasi-fractional approximations used here are determined by the simultaneous use of power series and asymptotic expansions [6–8]. Nevertheless, the form of the approximation is now defined by a combination of fractional powers and exponential functions in such a way that the singularities of the approximation reflect those of $J_\nu(x)$ at zero and infinity. Only first-degree polynomials have been used and all the parameters of the approximation are obtained as functions of the parameter ν.

For some values of ν we get coincidence with the exact function or with other previously published approximations [6, 9].

Better approximants have been published for $J_0(x)$ and $J_1(x)$, either by using Chebyshev polynomials combined with an algebraic mapping from $[-1, 1]$ to $[0, \infty]$ [10, 11] or by using our two-point quasi-fractional approximants [12]. However, the number of parameters that these approximants must use to improve the accuracy is much larger than the number presented here; also, the degrees of the polynomials used in [12] are greater than one. Moreover, the cases discussed in the Chebyshev-series paper [11] do not include the functions $J_\nu(x)$ with negative fractional ν. This is so because only cases which are "natural–natural" or "essential–natural" for the boundary conditions (see p. 69 of Ref. [11]) are considered there. On the other hand, the functions $J_\nu(x)$, ν fractional and negative, correspond to "essential–essential" cases due to the singularity at $x = 0$, as well as to the case due to the singularity at $x = \infty$ already considered there. Furthermore, the main practical disadvantage of the method described in Refs. [10, 11], compared with the method presented here, is that the calculation of the parameters for every ν has to be started afresh. In our method simple analytical expressions of the parameters as functions of ν are given.

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II. Procedure

The idea of the two-point quasi-fractional method is that both the approximated and the exact functions should have the same type of singularities at zero and at infinity. The function $J_\nu(x)$ has a branch point of order ν at $x=0$, and at infinity there is an essential singularity. The behaviour of this singularity for positive x is trigonometrical with a square root power factor. In order to obtain this behavior both at zero and at infinity we have to choose the form of the approximation as

$$\hat{J}_\nu(x) = \frac{x^\nu}{(1+x)^{\nu+1/2}} \left[\frac{P_0 + P_1 x}{q_0 + q_1 x} \cos x + \frac{p_0 + p_1 x}{q_0 + q_1 x} \sin x \right]. \tag{1}$$

This has the same kind of branch point at $x=0$ as $J_\nu(x)$ and the same form of asymptotic expansion at infinity. This is the simplest form of the approximation.

The parameter q_0 or the parameter q_1 is arbitrary and the other parameters (P_0, P_1, p_0, and p_1) are determined by using the power series and asymptotic expansions of $J_\nu(x)$.

$$J_\nu(x) = \begin{cases} \dfrac{x^\nu}{2^\nu \Gamma(\nu+1)} - \dfrac{x^{\nu+2}}{2^{\nu+2}\Gamma(\nu+2)} + \cdots; & x \ll 1 \qquad (2) \\[2ex] \sqrt{\dfrac{2}{\pi x}}\, [\cos(x-\alpha) + \cdots]; \quad x \gg 1; & \alpha = \dfrac{1}{2}\nu\pi + \dfrac{\pi}{4}. \qquad (3) \end{cases}$$

By expanding the power series and using the asymptotic expansions of Eq. (1) and by equating these expansions to those of Eqs. (2) and (3), we obtain the parameters of the approximation. The best procedure is to choose $q_1 = 1$ and to obtain P_1 and p_1 from the leading terms of the asymptotic expansion (3). Then, we multiply the power series (2) by the denominator of Eq. (1), and we equate the corresponding terms of the numerator after $\cos x$ and $\sin x$ have been replaced by their power series. By using the same denominator, $q_0 + q_1 x$, all the parameters are obtained by solving linear equations. The results are

$$\begin{aligned}
P_0 &= 4(1+\nu)\left(\frac{2\sin\alpha}{2\nu+1} - \frac{\sqrt{\pi/2}}{2^\nu\Gamma(\nu+1)}\right); \qquad \alpha = (2\nu+1)\frac{\pi}{4} \\
P_1 &= (2\nu^2+\nu+1)\cos\alpha \\
p_0 &= 4(1+\nu)\sin\alpha - (2\nu^2+\nu+1)\cos\alpha - \frac{(2\nu^2+5\nu+1)\sqrt{\pi/2}}{2^\nu\Gamma(\nu+1)} \\
p_1 &= (2\nu^2+\nu+1)\sin\alpha \\
q_0 &= 2^{\nu+2}(\nu+1)!\left[\frac{2\sin\alpha}{2\nu+1} - \frac{\sqrt{\pi/2}}{2^\nu\Gamma(\nu+1)}\right] \\
q_1 &= (2\nu^2+\nu+1)\sqrt{\pi/2}.
\end{aligned} \tag{4}$$

For $\nu = -\frac{1}{2}$, the fraction $(\sin\alpha)/(2\nu+1)$ tends to $\pi/4$. In this case the values of the parameters are

$$p_0 = p_1; \qquad P_0 = \pi - 2; \qquad P_1 = 1; \qquad q_0 = (\pi - 2)\sqrt{\pi/2}; \qquad q_1 = \sqrt{\pi/2}$$

and the approximation coincides with the exact function

$$\hat{J}_{-1/2}(x) = J_{-1/2}(x) = \sqrt{2/\pi x}\cos x. \tag{5}$$

For $\nu = 0$ we obtain the approximation previously published for $J_0(x)$ [6].
For $\nu = -1$, we obtain the approximation

$$\hat{J}_{-1}(x) = \frac{\sqrt{1+x}}{2+\sqrt{\pi x}}\left[\cos x - (1+x)\frac{\sin x}{x}\right], \tag{6}$$

which can also be considered an approximation to $-J_1(x)$, with a maximum error smaller than the error published in [9].

For any $\nu < 0.5$ the pole of the approximation occurs at $x < 0$ and therefore does not destroy the precision of the approximation on the positive axis. For $\nu = 0.5$ however we have the pole at $x = 0$, and for larger values of ν the pole is one the positive side of the real axis so the approximation fails. For this reason if we want an approximation for ν up to 1, we have to modify the form of the approximation in such a way that all the singularities are kept around $x = 0$ and $x = \infty$.

With these ideas in mind we have chosen the form of the approximation for ν around 1 as

$$\check{J}_\nu(x) = \frac{x^{\nu-1}}{(1+x)^{\nu-1/2}}\left[\frac{\bar{P}_1 x}{\bar{q}_0 + \bar{q}_1 x}\cos x + \frac{\bar{p}_0 + \bar{p}_1 x}{\bar{q}_0 + \bar{q}_1 x}\sin x\right]. \tag{7}$$

The parameters $\bar{P}_1$, $\bar{p}_0$, $\bar{p}_1$, $\bar{q}_0$, and $\bar{q}_1$ are obtained from the series expansions (2) and (3) in a manner similar to that used to obtain the parameters of $\hat{J}_\nu(x)$. The results are now

$$\begin{aligned} \bar{P}_1 &= \cos\alpha \\ \bar{p}_0 &= (\nu - \tfrac{1}{2})^{-1}\left(\sin\alpha - \frac{\sqrt{\pi/2}}{2^\nu \Gamma(\nu+1)}\right) - \cos\alpha \\ \bar{p}_1 &= \sin\alpha \\ \bar{q}_0 &= 2^\nu \nu!\,(\nu - \tfrac{1}{2})^{-1}\left(\sin\alpha - \frac{\sqrt{\pi/2}}{2^\nu\Gamma(\nu+1)}\right) \\ \bar{q}_1 &= \sqrt{\pi/2}. \end{aligned} \tag{8}$$

In the case of $\nu = \frac{1}{2}$, $\bar{P}_1$ is zero and for $\bar{p}_0$ and $\bar{q}_0$ it is convenient to find the limit of the expression

$$\lim_{\nu \to 1/2}\left\{(\nu - \tfrac{1}{2})^{-1}\left(\sin\alpha - \frac{\sqrt{\pi/2}}{2^\nu\Gamma(\nu+1)}\right)\right\} = 2 - \ln 2 - \gamma \simeq 0.73,$$

where γ is the Euler constant. Since this limit is finite, the approximation and the exact functions are coincident for $\nu = \frac{1}{2}$:

$$\check{J}_{1/2}(x) = \sqrt{2/\pi x}\,\sin x = J_{1/2}(x). \tag{9}$$

For $\nu = 1$, we obtain for $J_1(x)$ the approximation

$$\check{J}_1(x) = \frac{1}{\sqrt{1+x}}\left[\frac{-x}{2(2-\sqrt{\pi}) + \sqrt{\pi x}}\cos x + \frac{3-\sqrt{\pi}+x}{2(2-\sqrt{\pi})+\sqrt{\pi x}}\sin x\right]. \tag{10}$$

This can also be considered an approximation for $J_{-1}(x)$ and has been reported previously [9].

For ν in the interval (0, 1), the zeros of the denominator occur at negative values of x; therefore this approximation can be used in the interval (0, 1) throughout. For values of ν smaller than 0.5, the maximum error using $\hat{J}_\nu(x)$ is smaller than that using $\check{J}_\nu(x)$ and therefore it is more advantageous to use the first one.

III. Graphic Results

In Fig. 1 we show the maximum error of the approximations as a function of ν. In each case we have chosen the best approximation; therefore for $\nu \geqslant 0.5$ the approximation used is $\check{J}_\nu(x)$ and for $\nu \leqslant 0.5$ we have used $\hat{J}_\nu(x)$.

For negative values of ν, the function becomes infinite at $x = 0$. In order to avoid this situation and to obtain a better assesment of the error, we have computed $|x^{-\nu}[\hat{J}_\nu(x) - J_\nu(x)]|$ instead of $|\hat{J}_\nu(x) - J_\nu(x)|$. Only for the case $\nu = -1$ can both methods of computing the error be used; this is shown by the isolated point at $\nu = -1$, together with the continuous curve. The variation of the error with the independent variable x is shown in Figs. 2 and 3. In Fig. 2 only negative values of

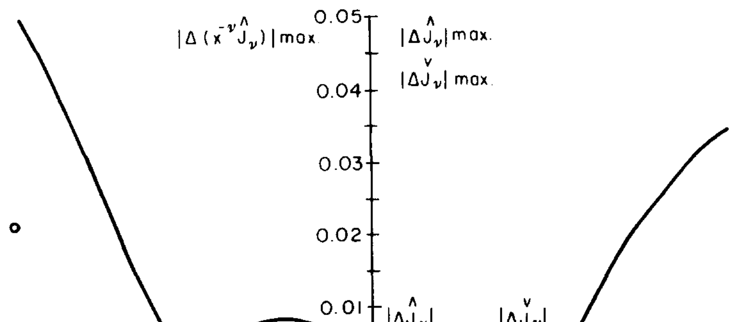

FIG. 1. Maximum errors as a function of ν. For negative ν the error shown corresponds to $x^{-\nu}\hat{J}_\nu$. For $0 \leqslant \nu < 0.5$ and $\nu \geqslant 0.5$ the errors are those of $\hat{J}_\nu$ and $\check{J}_\nu$, respectively (note the seprator mark).

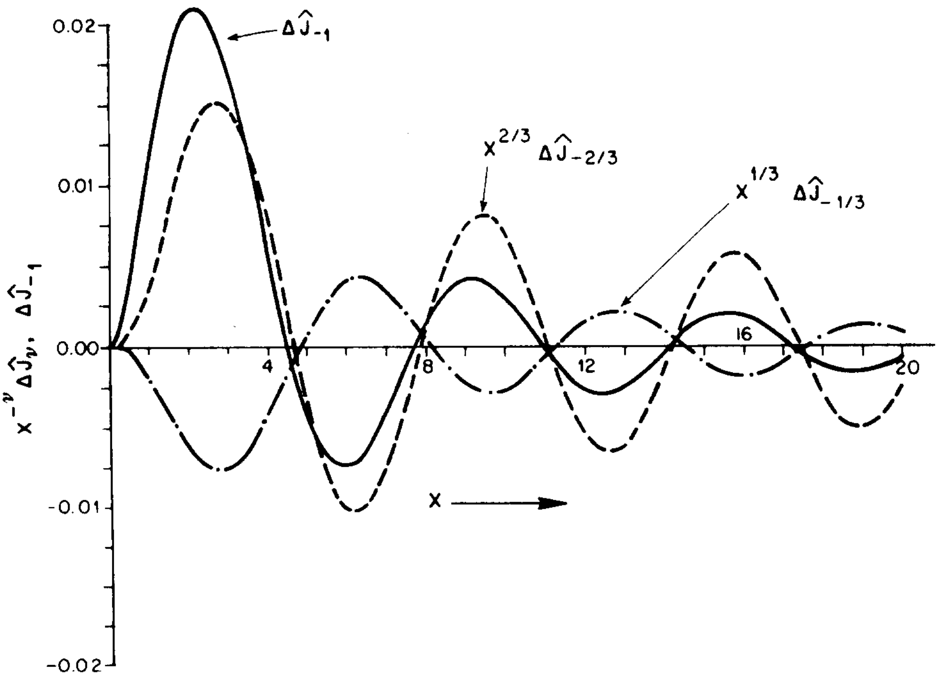

FIG. 2. Errors of the functions $\hat{J}_{-1}$, $x^{2/3}\hat{J}_{-2/3}$, and $x^{1/3}\hat{J}_{-1/3}$ as functions of x.

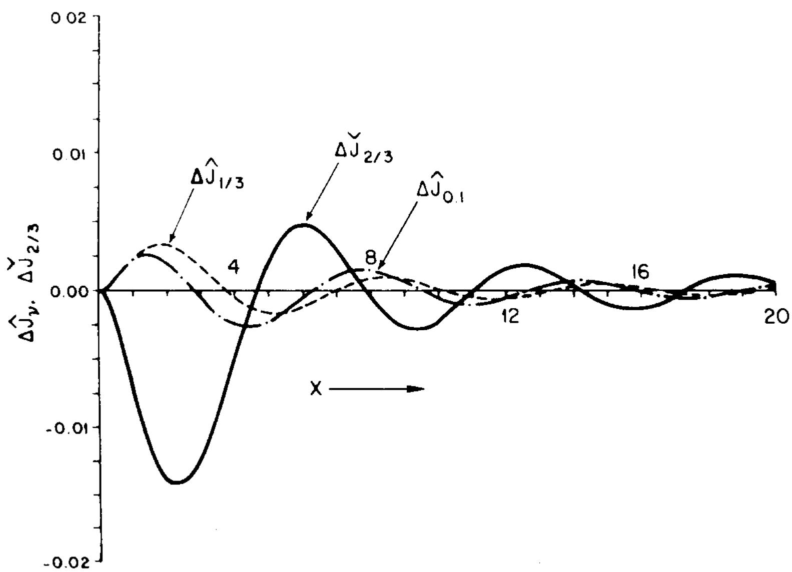

FIG. 3. Errors of the functions $\hat{J}_{0.1}$, $\hat{J}_{1/3}$, and $\check{J}_{2/3}$ as functions of x.

ν have been chosen $(-1, -\frac{2}{3}, -\frac{1}{3})$ as representative of the general behaviour. The errors are very small around zero and for large values of x; the maximum errors occur at x between 1 and 5. In the case of positive ν (Fig. 3) we have selected the values $\nu=0.1$, $\frac{1}{3}$, and $\frac{2}{3}$. For $\nu=0.1$ and $\nu=\frac{1}{3}$ we use $\hat{J}_\nu(x)$ and for $\nu=\frac{2}{3}$ we use $\check{J}_\nu(x)$. The same pattern for negative ν appears for positive ν.

The approximations and the exact functions are coincident for ν equal to -0.5 and 0.5.

IV. Conclusions

A simple approximation to the Bessel functions has been found that gives at least two-digit precision for the full range of the variable $x \geqslant 0$ and for $|\nu| < 0.6$. The maximal error is about 0.3 for $|\nu|$ near one.

This approximation uses only rational functions of first degree combined with fractional powers and trigonometric functions. We deem the precision obtained to be sufficient for a great number of applications in which these functions appear. The results for integer ν coincide with those reviously published and for $\nu = \pm\frac{1}{2}$ the approximations coincide with the exact functions.

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